Theorem 2: Given a constant matrix A[a constant part of $S(t)$], there exists a positive matrix P such that

i)

$$\delta=\min \delta_1,\delta_2 \mid \delta_i=\min_{\alpha_i=0-2\pi} M(S(t))$$

ii)

$M(B)\leq 0$ for all matrix $S(t)$ continuously with

$$\|S(t)-A\|<\delta \text{ for all } t\geq 0$$

$$B\overset{\text{def}}{=}A+(\delta/2)I$$

then the trivial solution of Eq. (13) is stable in the sense of Liapunov.

Proof: Since the real parts of the eigenvalues of

$$B\overset{\text{def}}{=}A-[\tfrac{1}{2}\delta+M(S(t))]I$$

are all negative or zero, then there exists a positive definite Hermitian matrix P such that

$$PB+B^TP=-C(C\geq 0)$$

The eigenvalues of

$$PS(t)+S^T(t)P+\delta P\overset{\text{def}}{=}L$$

vary continuously with $S(t)$ and as $S(t)\to A$, the matrix L tends to $PA+A^TP-\{\delta+2M[S(t)]\}P=PB+B^TP=-C$. Therefore the matrix L is seminegative definite throughout some neighborhood $\|S(t)-A\|<\delta$ of A.

Now define a Liapunov function

$$V(X)=X^TPX;\ P=P^T>0$$

then $\dot{V}$ along the trajectories of Eq. (14) yields:

$$\dot{V}(X)=X^T[PS(t)+S^T(t)P]X\leq X^T[\delta+2M(S(t))]PX$$

$$=-\delta X^TPX=-\delta V<0$$

The rest of the arguments are standard and so omitted. This completes the proof.

The prototype AMCD spacecraft was found to be asymptotically stable where the parameters used are: $I_a=680$ kg$-$m^2, $I_{az}=1360$ kg$-$m^2, $I_s=680$ kg$-$m^2, $I_{sz}=453.3$ kg$-$m^2, $a=0.76$ m, $a'=0.76$ m, $\omega_{\alpha 0}=401.3$ rad/sec, $\omega_{\beta 0}=0.1$ rad/sec, $k_\phi=1020$, $k_{\dot\phi}=2856$. Simulation was reported for larger k_ϕ, $k_{\dot\phi}$, where the system was found stable. A considerable amount of work is needed for selection of the best parameters. Further simulation is needed when one or more ball-in-tube dampers are present. Note that smaller damper masses appear to stabilize the system.

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Fourth-Order Runge-Kutta Integration with Stepsize Control

David G. Hull*
University of Texas at Austin, Austin, Texas

Introduction

An engineer wanting to integrate a system of ordinary differential equations has at his disposal a wide variety of numerical integration methods. Included are single-step (Runge-Kutta) integrators up to eighth order[1,2] and multistep (Adams-Bashforth, Adams-Moulton) integrators of any order.[3] Furthermore, these integrators are available in both fixed-step and variable-step versions. In spite of this, however, the fixed-step, fourth-order Runge-Kutta method is used with a degree of regularity even though the problem being solved does not require fixed-step integration. Perhaps the reason for this is that engineers tend to use what has worked in the past and do not always have time to experiment with numerical methods.

It is well known that variable-step integration is more efficient than fixed-step integration; that is, fewer integration steps are required to achieve the same accuracy. Hence, the purpose of this Note is to present a simple step-size control procedure for the classical fourth-order Runge-Kutta method. Since the implementation of the step-size procedure requires only about 15 FORTRAN statements, anyone using the fixed-step version can make the conversion quite easily.

Step-size control already is available for the fourth-order Runge-Kutta method.[4] One method, doubling, uses two regular steps and a simultaneous double step to estimate the truncation error of the fourth-order method. It requires five and one-half function evaluations per integration step and costs seven function evaluations for a rejected step. Another method uses a sequence of accepted steps to predict the next step. However, this approach requires additional storage, and the single-step character of the Runge-Kutta method is destroyed.

The step-size control procedure being proposed here uses the concept of embedded methods. Here, a lower-order method using the same function evaluations as the fourth-order method is isolated. Because of the nature of the function evaluations needed to generate the fourth-order method, a third-order method cannot be found, and a second-order method must be used, as indicated by Sarafyan (Ref. 5, p. 71). The difference of the solutions obtained by the second- and fourth-order methods is used as an estimate of the truncation error of the second-order method. This approximate truncation error then is used to find the size of the integration step which maintains a prescribed relative error.

Fixed-Step Integration

The fourth-order Runge-Kutta method is to approximate the solution of the initial-value problem

$$\frac{dx}{dt}=f(t,x),\quad x_0=x(t_0) \tag{1}$$

by the relation

$$\hat{x}(t_0+h)=x_0+h\sum_{k=0}^{3}\hat{c}_kf_k \tag{2}$$

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*Associate Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.

where the four function evaluations are given by

$$f_0 = f(t_0, x_0) \tag{3a}$$

$$f_k = f\left(t_0 + h\alpha_k,\ x_0 + h\sum_{\lambda=0}^{k-1} \beta_{k\lambda} f_\lambda\right),\quad k>0 \tag{3b}$$

In these relations, x and f are vectors, and h is the step size. The constants α, β, $\hat{c}$ are obtained from the equations of condition (see, for example, Ref. 1) which result from expanding $x(t_0+h)$ and Eq. (2) in Taylor series (assuming h small) and equating coefficients of similar terms through $0(h^4)$.

The system of equations of condition has two free parameters that can be used to create a fourth-order method having minimum truncation error.[6] However, for the standard Runge-Kutta method, two additional conditions are imposed, that is, $\alpha_1 = \alpha_2$ and $\hat{c}_1 = \hat{c}_2$. Then, the solution of the equations of condition can be found and is presented in Table 1.

Variable-Step Integration

A second-order method can be obtained by using two of the four function evaluations of the fourth-order method. Here, however, three function evaluations are used to find the second-order method of minimum truncation error, the philosophy being that the smaller truncation error will lead to the prediction of larger integration steps.

The second-order appproximation is defined by

$$x(t_0+h) = x_0 + h\sum_{k=0}^{2} c_k f_k \tag{4}$$

where the constants α, β defining the three function evaluations are those of the fourth-order method (Table 1). The use of this approximation leads to equations of condition whose solution is given by

$$c_0 = 0,\quad c_1 + c_2 = 1 \tag{5}$$

and contains one free parameter. The free parameter now is chosen to make the second-order method as close to a third-order method as possible. This is done by minimizing the norm

$$\|T\| = (T_1^2 + T_2^2)^{1/2} \tag{6}$$

where

$$T_1 = c_2/4 - 1/6,\quad T_2 = (c_1 + c_2)/8 - 1/6 \tag{7}$$

are the third-order equations of condition which cannot be satisfied because there are not enough parameters. In view of Eq. (5), it is obvious that $\|T\|$ is a minimum when $c_2 = 2/3$, so that $c_1 = 1/3$. The coefficients are summarized in Table 1.

The difference $\hat{x} - x$ of the fourth-order and second-order approximations is used as an estimate of the truncation error vector of the second-order method, that is

$$TE = h\sum_{k=0}^{3} \delta_k f_k,\quad \delta_k = \hat{c}_k - c_k \tag{8}$$

Table 1 Coefficients for the Runge-Kutta 2(4)

k	α_k	$\beta_{k\lambda}$ $\lambda=0$	$\lambda=1$	$\lambda=2$	$\hat{c}_k$	c_k	δ_k
0	0	...	...	...	1/6	0	1/6
1	1/2	1/2	...	...	1/3	1/3	0
2	1/2	0	1/2	...	1/3	2/3	−1/3
3	1	0	0	1	1/6		1/6

and the coefficients δ are listed in Table 1. Next, the relationship between the truncation error and the step size is given by

$$TE = \dddot{x}_0 h^3/3! \tag{9}$$

where the right-hand side is the third-order term of the Taylor series expansion of $x(t_0+h)$.

The procedure for predicting the step size is straightforward. The size of the first integration step is prescribed, and the trunction error of each variable is computed at the end of the step using Eq. (8). The variable having the maximum relative truncation error is identified (subscript m), and the maximum relative truncation error is written as

$$RTE_m = TE_m/x_m = \dddot{x}_{m0} h^3/(3!x_m) \tag{10}$$

This relation is used to compute $\dddot{x}_{m0}$ from the known values of RTE_m, h, and x_m. Then, with $\dddot{x}_{m0}$ assumed constant, Eq. (10) is used to find the value of h which would make the relative truncation error equal to a prescribed integration tolerance. The desired relation is

$$h_2 = h_1 (TOL\ x_{m2}/RTE_m x_{m1})^{1/3} \tag{11}$$

where the subscript 1 denotes the first step, and subscript 2 denotes the improved step. The quantity x_{m2} is not known but can be determined iteratively. In practice, however, the results obtained using Eq. (11) differ very little from those obtained by assuming $x_{m2} = x_{m1}$ in computing h_2. Hence, the recommended formula for computing h_2 is

$$h_2 = h_1 (TOL/RTE_m)^{1/3} \tag{12}$$

At this point, if $h_2 > h_1$, the integration step h_1 is accepted, and h_2 is used for the next integration step. On the other hand, if $h_2 < h_1$, the step h_1 is rejected, and the integration step is repeated using h_2. Finally, to minimize the number of rejected steps, only a fraction (say 0.9) of the step predicted by Eq. (12) is used.

Example

The performance of the step-size formula (12) has been compared with that of doubling. The results for integrating $\dot{x} = -x$ from $x(0) = 1$ to $t = 20$ are shown in Table 2, where $TOL = 10^{-k}$ and where the numbers for doubling have been taken from Ref. 7, p. 82. A log-log plot of the data shows that doubling requires approximately 13% more function evaluulation (FE) at $TOL = 10^{-3}$ and 23% more at $TOL = 10^{-9}$. Also, Table 2 shows that the relative error (RE) achieved is the same order of magnitude as that requested, whereas the same is not true for doubling. It is emphasized that results are valid only for the example problem, and that many more problems must be solved before any general conclusions can be drawn.

Table 2 Comparison of two step-size estimators

k	Eq. (12) Re	FE	Doubling[7] RE	FE
3	2.7 E-3	250	6.8 E-3	187
4	2.8 E-4	422	2.2 E-3	286
5	2.8 E-5	726	3.9 E-4	440
6	2.9 E-6	1270	6.2 E-5	704
7	2.9 E-7	2230	9.8 E-6	1100
8	2.9 E-8	3942	1.6 E-6	1738
9	2.9 E-9	6986	2.5 E-7	2728
10			3.9 E-8	4367
11			6.2 E-9	6798

Discussion and Conclusions

A step-size control procedure based on an estimate of the truncation error of an embedded second-order method has been presented for the classical fourth-order Runge-Kutta method. For the sample problem, this procedure has been shown to be more efficient than the method of doubling as applied by Gear.[7]

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Some Laser Velocimeter Measurements in the Turbulent Wake of a Supersonic Jet

Ronald D. Flack Jr.*
University of Virginia, Charlottesville, Va.
and
H. Doyle Thompson†
Purdue University, West Lafayette, Ind.

Introduction

ALTHOUGH the phenomena of turbulent jet wakes have been studied for a long time, many fundamental questions have not been answered. A large portion of turbulent wake data has been obtained by pitot static probes or by hot-wire anemometers. Unfortunately, for high subsonic, transonic, and supersonic flow, hot-wire anemometers and pitot probes are difficult to use. The laser velocimeter (LV) has been publicized as one remedy to these difficulties.[1,2]

Previous examinations of turbulent jet wakes were directed primarily toward subsonic flows.[3,4] Eggers[5] and others have measured supersonic jet wakes with pitot probes. Conventional hot-wire data are available for most of the subsonic flows, including energy spectra, spatial correlations, and turbulent shear stresses. However, for the supersonic flows only mean axial velocities previously were available. More recently, Avidor[6] and others used laser velocimeters to measure the turbulent fields of compressible jets. These LV investigations, however, were not aimed at the flow data but were meant to be investigations of laser velocimeters. As a result, complete turbulence data and correlations are not available.

In the present study, the near wake of a supersonic jet with a Mach number of 2.22 is examined with a LV. Mean flow data are found and compared to previously obtained supersonic pitot probe data. Turbulence intensities are found and compared to previously obtained subsonic hot-wire values. Differences between the present LV data and previous data are discussed. The purpose of this Note is therefore twofold. First, potential sources of previously unaddressed LV biases are brought to the attention of other experimenters. Second, unbiased data are presented for a portion of supersonic jet wake where Mach number effects are important.

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*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

†Professor, School of Mechanical Engineering. Member AIAA.

Experimental Apparatus and Procedure

The experimental investigation was conducted utilizing an air blowdown system and a converging-diverging axisymmetric nozzle with an exit diameter ($D=2r_e$) of 25.78 mm and a designed exit Mach number of 2.22.[5,7] The system was operated at a stagnation pressure in a plenum tank which was chosen to match the jet exit pressure to the ambient atmospheric pressure. The LV used was of the individual realization dual-beam forward scatter type. A 5-W Coherent argon-ion laser was used for a one-component system. The effective probe volume size was approximately 176 μ diam and approximately 1500 μ long. Frequency shifting by the use of a Bragg cell or other technique was not used in the present investigation. The entire optical system was mounted on a large mill table capable of traversing the flow in three directions, and the positioning accuracy was better than 0.1 mm.

As a particle passes through the probe volume, light is scattered, collected by the receiving optics, and focused onto a photomultiplier tube (PMT). After the low-frequency pedestal voltage was removed from each PMT signal by the use of a bandpass filter, each Doppler signal was analyzed by a zero-crossing burst-type four- and eight-count comparison processor.[7] Dioctyl phthalate (DOP) particles 0.5-2 μ in diameter were generated with a Laskin nozzle with impactor plates. In previous inviscid steady internal transonic flow measurements,[8] the LV was found to operate very accurately. The average velocity LV data generally were within 2% of static pressure data, and the LV signal processor was found to operate with a very low noise level (measured "turbulence intensities" less than 0.01).

With an individual realization laser velocimeter, such as the one utilized in the present study, the output of the processor is not continuous. Therefore, if the sample rate is not high enough to insure that a significant number of samples are being measured within a temporal interval less than the macro time scale, obtaining energy spectrum data or length or time-scale data, as with a hot-wire anemometer, is not possible. Smith and Meadows[9] and Mayo et al.[10] have demonstrated that with high enough sample rates obtaining spectral data is possible. However, with high-speed flows as presented in this Note, seeding the flow densely enough is not possible to obtain this high sample rate. Thus, the signals were treated in velocity histograms, with other quantities of interest calculated therefrom, including the mean velocity and turbulence intensity. Only the near-wake region of the supersonic turbulent jet was "probed" in the present investigation. Axial velocity data were obtained for five axial positions X. At each axial position, velocity histograms were recorded for 15 to 20 radial positions r. Approximately 1500 data samples were obtained for each position.

Results and Discussion

Laser Velocimeter Biases

Data from the velocity histograms have been analyzed and used to construct mean velocity profiles at each axial location. A typical plot is presented in Fig. 1 for $X/D=1.005$. Com-